SECONDARY FLOWS IN A ROTATING CIRCULAR CYLINDER OF INCOMPRESSIBLE CONDUCTING FLUID AS THE RESULT OF THE SUDDEN TURNING-ON OF A TRANSVERSE MAGNETIC FIELD

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UDC 538.4

We study vortex flows in a rotating circular cylinder of incompressible fluid resulting from the sudden turning-on of a transverse magnetic field. The investigation is performed for the initial stage when secondary flow is determined mainly by Lorentz forces, and the effect of viscosity and the convective transport of vorticity by secondary flow is negligibly small. No restrictions are imposed on the magnetic Reynolds number R_m for the basic rotational motion; the number R_m' calculated from a typical secondary flow speed is assumed small.

1. The magnetohydrodynamics of a rotating fluid has interesting applications in astrophysics and geophysics [1]. Most of the problems involved are related to nonstationary interactions of a rotating conducting fluid with a magnetic field.

The present paper is devoted to a study of secondary flows produced in a rotating fluid by a change in the external magnetic field. An idealized model of the rotating fluid is employed, and it is assumed that the magnetic field is turned on "instantaneously." The results obtained may be useful in the study of laboratory models of astrophysical and geophysical phenomena.

The method of solution and the simplifying assumptions are similar to those used by Sneyd [2].

Sneyd [2] treats vortex flow in a long circular cylinder of conducting fluid resulting from the sudden inclusion or ejection of an external magnetic field. The diffusion of the field into the fluid during the change of the external magnetic field produces an electric current in the fluid and Lorentz forces. If the Lorentz forces are nonpotential they cannot be balanced by a pressure gradient and lead to the production of vortex flow. Sneyd [2] considers simple geometry leading to nonpotential Lorentz forces – geometry with a uniform external field at right angles to the axis of the cylinder.

It is shown [2] that if the magnetic Reynolds number is small, it is possible to distinguish an initial stage in the development of vortex flow in the cylinder which lasts for a time of the order $t_0 = 4\pi\sigma a^2/c^2$, where a is the radius of the cylinder, and σ is the conductivity of the fluid. During this stage the flow is determined solely by the Lorentz forces, and other factors such as viscosity and convective transport of eddies have a negligible effect; they become important later. Sneyd [2] calculates the flow established to the end of this initial stage.

It is of interest to examine vortex flow resulting from the penetration of an external field into a moving fluid. In this case the magnetic field is established in the fluid both by diffusion and by the convective transport of lines of force. This complicates the distribution of Lorentz forces and the distribution of eddies in the fluid.

In the present paper the initiation of secondary vortex flow in a moving fluid is treated through the example of a rotating fluid cylinder. The conducting fluid is contained in a long nonconducting cylinder of

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhanika i Tekhnicheskoi Fiziki, No. 5, pp. 28-36, September-October, 1974. Original article submitted March 15, 1974.

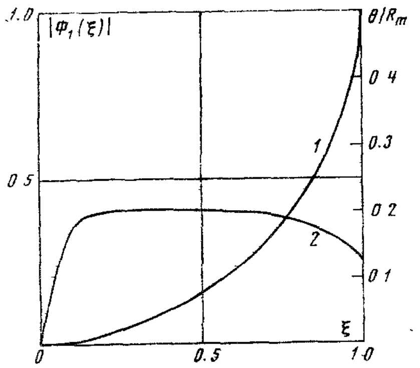

Fig. 1

inside radius a. The container and fluid rotate about the axis of the cylinder with a constant angular velocity Ω_0. At time $t=0$ a uniform external magnetic field is turned on instantaneously at right angles to the axis of the cylinder. It is required to determine the distribution of the magnetic field and the Lorentz forces inside the fluid, and from these to find the velocity distribution of secondary flow.

For $\Omega_0=0$ the problem under discussion reduces to Sneyd's problem [2], and therefore the solution presented below is a generalization of [2] to the case of a rotating cylinder. Just as in [2], the magnetic Reynolds number, calculated from a typical secondary flow of speed V, is assumed small

$$R_m' = 4\pi\sigma\, V\, a\, /\, c^2 \ll 1 \tag{1.1}$$

where the prime emphasizes that the number is calculated from the secondary flow speed.

No restrictions are imposed on the magnetic Reynolds number $R_m = 4\pi\sigma\Omega_0 a^2/c^2$ calculated from a typical velocity $a\Omega_0$ of the basic rotational motion. Since $R_m = 2\pi t_0/T$, where $t_0 = 4\pi\sigma a^2/c^2$ is the time of penetration of the field, and $T = 2\pi/\Omega_0$ is the time for one revolution of the cylinder, for $R_m \ll 1$ the rotation does not affect the penetration of the field or the character of the secondary flow. For $R_m \gtrsim 1$ this effect is significant, and the character of the secondary flow differs sharply from the case of a stationary cylinder.

As in [2] we do not consider the whole penetration process and the subsequent change of vortex flow. The purpose of the study is to obtain a picture of the flow up to the instant of almost complete establishment of a stationary magnetic field in the cylinder, i.e., up to the end of the initial stage. While this flow is steady in [2], in the case of a rotating cylinder the Lorentz forces do not vanish with the establishment of a stationary distribution of the magnetic field, and their effect on the secondary flow continues. We arbitrarily take the time $t=t_0$ as the instant of establishing the field. Most of the graphical material presented below refers to this time, although the solution obtained is valid for times somewhat longer than t_0, until the effect of viscosity and convective transport of eddies becomes significant.

2. We calculate the distribution of the magnetic field. In a cylindrical coordinate system (r, α, z) with the angle α measured from the direction of the applied uniform field H_0, the required magnetic field H has two components $[H_r(r, \alpha, t), H_\alpha(r, \alpha, t)]$ determined by the vector potential $\mathbf{A} = A(r, \alpha, t)\mathbf{e}_z$

$$\mathbf{H} = \operatorname{curl} \mathbf{A} = \left(\frac{1}{r}\frac{\partial A}{\partial \alpha}, -\frac{\partial A}{\partial r}, 0\right)$$

The change in the magnetic field as it penetrates the rotating cylinder is described by the equations

$$\Delta A_1 = 0, \quad \frac{\partial A_2}{\partial t} = -\Omega_0 \frac{\partial A_2}{\partial \alpha} + \nu_m \Delta A_2 \tag{2.1}$$

Here $\nu_m = c^2/4\pi\sigma$, and the subscripts 1 and 2 refer to the field outside and inside the cylinder, respectively. As the velocity in the convective term in the second equation we take the velocity $\mathbf{v}_0 = \Omega_0 r \mathbf{e}_\alpha$ of the basic rotational motion. On the basis of (1.1) the secondary flow does not affect the penetration of the field. The initial and boundary conditions are the same as in [2] for a stationary cylinder:

$$A_1(r, \alpha, 0) = H_0\left(r - \frac{a^2}{r}\right)\sin\alpha, \quad A_2(r, \alpha, 0) = 0 \tag{2.2}$$

$$A_1(a, \alpha, t) = A_2(a, \alpha, t), \quad \left.\frac{\partial A_1}{\partial r}\right|_{r=a} = \left.\frac{\partial A_2}{\partial r}\right|_{r=a} \tag{2.3}$$

$$A_1(r, \alpha, t) \to H_0 r \sin\alpha \quad \text{for} \quad r \to \infty \tag{2.4}$$

We seek a solution in the form

$$A_1(r, \alpha, t) = f_1(r, t)\, e^{i\alpha}, \quad A_2(r, \alpha, t) = f_2(r, t)\, e^{i\alpha}$$

For the complex functions $f_1(r, t)$ and $f_2(r, t)$ we obtain the equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_1}{\partial r}\right) - \frac{1}{r^2} f_1(r, t) = 0$$
$$\frac{\partial f_2}{\partial t} = \frac{\nu_m}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_2}{\partial r}\right) - \left(\frac{\nu_m}{r^2} + i\Omega_0\right) f_2(r, t) \tag{2.5}$$

$$f_1(r, 0) = -iH_0\left(r - \frac{a^2}{r}\right), \quad f_2(r, 0) = 0 \tag{2.6}$$

$$f_1(a, t) = f_2(a, t), \quad \frac{\partial f_1}{\partial r}\bigg|_{r=a} = \frac{\partial f_2}{\partial r}\bigg|_{r=a} \tag{2.7}$$

$$f_1(r, t)_{r\to\infty} \to -iH_0 r \tag{2.8}$$

The equations can be solved by taking Laplace transforms. The expressions for $A_1(r, \alpha, t)$ and $A_2(r, \alpha, t)$ have the form

$$A_1(r, \alpha, t) = -iH_0 re^{i\alpha}\left\{1 - \frac{a^2}{r^2}\left[1 + 2i\frac{J_1(i\beta)}{\beta J_0(i\beta)} + 4\sum_{n=1}^{\infty}\frac{e^{-(\alpha_n^2+iR_m)\tau}}{\alpha_n^2 + iR_m}\right]\right\} \tag{2.9}$$

$$A_2(r, \alpha, t) = 2H_0 ae^{i\alpha}\left\{-\frac{J_1(i\beta\xi)}{\beta J_0(i\beta)} + 2i\sum_{n=1}^{\infty}\frac{J_1(\alpha_n\xi)\, e^{-(\alpha_n^2+iR_m)\tau}}{J_1(\alpha_n)(\alpha_n^2 + iR_m)}\right\} \tag{2.10}$$

Here $\beta = \sqrt{iR_m}$ is a complex number, the α_n are the positive zeros of $J_0(x)$, $\tau = (\nu_m/a^2)t = t/t_0$ is the dimensionless time, and $\xi = r/a$ is the dimensionless radius. In the limiting case $\beta = 0(\Omega_0 = 0)$ the real part of (2.10) is the same as the result in [2].

For $\tau > 1$, i.e., after a lapse of time of the order t_0, the series in (2.9) and (2.10) practically vanish, and the solution becomes stationary. The steady magnetic field in this case differs from the uniform field H_0 since the lines of force are twisted by the rotating cylinder.

Pictures of the lines of force corresponding to the solution obtained are not presented in graphical form, since a graphical representation of them can be obtained from illustrative materials presented in [3], where a numerical method is used to investigate the behavior of the magnetic field in a rotating conducting fluid.

The difference between the initial conditions for the field taken in [3] and those in the present paper is unimportant for establishing the pattern of the magnetic field distribution. The difference in boundary conditions is important. In the present paper the rotating cylinder is in a nonconducting medium, and the natural boundary condition at infinity is used in the solution. In [3] it was assumed that outside the rotating fluid of finite conductivity σ there is a boundary with a stationary perfectly conducting fluid. This condition leads to the very strong increase of the magnetic field at the outer boundary of the rotating fluid noted in [3] for large R_m. In the problem under discussion the steady magnetic field outside the cylinder as $R_m \to \infty$ coincides with the initial distribution and the maximum value of the field is $2H_0$ (at $\alpha = \pm\pi/2$); i.e., the field strength is doubled.

The ejection of the magnetic field from the rotating fluid for large values of R_m is unrelated to the boundary conditions employed.

We calculate the eddy velocity distribution. In a moving conducting fluid the change in vorticity $\mathbf{w} = \mathbf{curl\,v}$ occurs both as a result of convective transport and diffusion because of viscosity, and as a result of the nonpotential Lorentz force c^{-1} $[\mathbf{j} \times \mathbf{H}]$. For two-dimensional flow of an incompressible fluid the equation for the vorticity has the form

$$\frac{\partial\omega}{\partial t} + (\mathbf{v}_0\nabla)\,\omega + (\mathbf{v}\nabla)\,\omega = \frac{1}{\rho c}\mathrm{curl}[\mathbf{j} \times \mathbf{H}] + \nu\Delta\omega \tag{3.1}$$

Here ν is the kinematic viscosity, $\mathbf{v} = \Omega_0 r\mathbf{e}_\alpha$ is the given velocity of rotational motion, and $\mathbf{v}$ denotes the secondary flow velocity.

It is shown in [2] that the viscosity and the convective term (in our case one of two convective terms) do not affect the distribution of vorticity established up to the time t_0 of the total penetration of the magnetic field into the cylinder.

As is clear from the estimate

$$\left|\frac{(\mathbf{v}\nabla)\,\omega}{\partial\omega/\partial t}\right| \sim \frac{Va}{\nu_m} = R_m', \qquad \left|\frac{\nu\Delta\omega}{\partial\omega/\partial t}\right| \sim \frac{\nu}{\nu_m} \tag{3.2}$$

assumptions (1.1), and the inequality $\nu/\nu_m \ll 1$, which holds for liquid metals and electrolytes, the last terms on both sides of Eq. (3.1) can be neglected in comparison with the first term.

As a result we obtain the equation

$$\frac{\partial\omega}{\partial t} + \Omega_0 \frac{\partial\omega}{\partial\alpha} = \frac{1}{\rho c}\mathrm{curl}_z[\mathbf{j} \times \mathbf{H}] \quad (\omega = \omega \mathbf{e}_z) \tag{3.3}$$

which differs from the corresponding equation in [2] only by the convective term $\Omega_0\, d\omega/d\alpha$ related to the basic rotational motion. The right-hand side of Eq. (3.3) is evaluated from the known solution (2.10) for the magnetic field. Calculations give

$$\frac{1}{\rho c}\mathrm{curl}_z[\mathbf{j} \times \mathbf{H}] = \mathrm{Re}\,\{F_1(r, t)\, e^{2i\alpha} + F_2(r, t)\} \tag{3.4}$$

$$F_1(r, t) = \frac{H_0^2}{\pi\rho a^2}\frac{1}{\xi}\left\{\frac{1}{\beta J_0(i\beta)}\sum_{n=1}^{\infty}\frac{[J_1(i\beta\xi)]^2}{J_1(\alpha_n)}\frac{d}{d\xi}\left[\frac{J_1(\alpha_n\xi)}{J_1(i\beta\xi)}\right]e^{-(\alpha_n^2+iR_m)\tau}\right. \tag{3.5}$$

$$\left. - 2i\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{J_1(\alpha_n\xi)\,\alpha_k J_1'(\alpha_k\xi)\, e^{-(\alpha_n^2+\alpha_k^2+2iR_m)\tau}(\alpha_k^2-\alpha_n^2)}{J_1(\alpha_n)\, J_1(\alpha_k)\,(\alpha_n^2+iR_m)(\alpha_k^2+iR_m)}\right\}$$

$$F_2(r, t) = \frac{H_0^2}{\pi\rho a^2}\frac{1}{\xi}\left\{-\frac{1}{\beta J_0(i\beta)}\sum_{n=1}^{\infty}\frac{(d/d\xi)\,[J_1(\alpha_n\xi)\, J_1(i\beta\xi)]}{J_1(\alpha_n)}\right. \tag{3.6}$$

$$\times \frac{\alpha_n^2+iR_m}{\alpha_n^2-iR_m}e^{-(\alpha_n^2-iR_m)\tau} - iR_m\left(\frac{J_1(i\beta\xi)}{\beta J_0(i\beta)}\right)^*\frac{J_1'(i\beta\xi)}{J_0(i\beta)} +$$

$$\left. + 2i\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{J_1(\alpha_n\xi)\,\alpha_k J_1'(\alpha_k\xi)\,(\alpha_n^2-\alpha_k^2)\, e^{-(\alpha_n^2+\alpha_k^2)\tau}}{J_1(\alpha_n)\, J_1(\alpha_k)\,(\alpha_n^2-iR_m)(\alpha_k^2+iR_m)}\right\}$$

The complex functions $F_1(r, t)$ and $F_2(r, t)$ do not depend on the angle α, and therefore the α dependence is described solely by the factor $\exp(2i\alpha)$ in the first term of (3.4). Because of this factor the vortex flow degenerates into four separate cells when $\Omega_0 = 0$. The second term of $F_2(r, t)$ is related to the rotation and is absent when the cylinder is stationary.

As a result of the presence of the factor $\exp(-\alpha_n^2\tau)$ in Eqs. (3.5) and (3.6) the strong action of the magnetic field on the rotating fluid,

$$\frac{1}{\rho c}\mathrm{curl}_z[\mathbf{j} \times \mathbf{H}] = \mathrm{Re}\left\{iR_m\frac{H_0^2}{\pi\rho a^2}\frac{1}{\xi}\left(\frac{J_1(i\beta\xi)}{\beta J_0(i\beta)}\right)^*\frac{J_1'(i\beta\xi)}{J_0(i\beta)}\right\} \tag{3.7}$$

ceases to depend on the angle α and the time t when $\tau \gg 1$. Here the asterisk denotes the complex conjugate. This quantity together with the viscosity determines the steady flow which was investigated in [4] for $R_m \gg 1$.

The solution of Eq. (3.3) with the right-hand side of (3.4) satisfying the initial condition $\omega(r, \alpha, 0) = 2\Omega_0$ is sought in the complex form:

$$\omega(r, \alpha, t) = 2\Omega_0 + \omega_1(r, \alpha, t) + \omega_2(r, t) \quad \omega_1 = \varphi(r, t)\, e^{2i\alpha} \tag{3.8}$$

The unknowns $\varphi(r, t)$ and $\omega_2(r, t)$ satisfy zero initial conditions and the equations

$$d\varphi/dt + 2i\Omega_0\varphi = F_1(r, t), \quad d\omega_2/dt = F_2(r, t)$$

Omitting the intermediate calculations, including the summing of the series by using contour integration, the result of the integration can be written in the form

$$\omega_1(r, \alpha, t > t_0) = \frac{H_0^2 e^{2i(\alpha-\Omega_0 t)}}{4\pi\rho\nu_m}\Phi_1(\xi), \quad \omega_2(r, t > t_0) = \frac{H_0^2}{4\pi\rho\nu_m}\Phi_2(\xi, \tau) \tag{3.9}$$

$$\Phi_1(\xi) = \frac{2i}{\xi}\left\{\frac{J_2(i\beta\xi)\, J_1(\beta\xi)}{\beta J_0(i\beta)\, J_0(\beta)} - 4\sum_{n=1}^{\infty}\frac{J_2(\alpha_n\xi)\, I_1(\alpha_n\xi)\,\alpha_n^2}{I_0(\alpha_n)\, J_1(\alpha_n)\,(\alpha_n^4+R_m^2)}\right\} \tag{3.10}$$

$$\Phi_2(\xi, \tau) = -\frac{4}{\xi}\left\{\frac{J_1(i\beta\xi)}{\beta J_0(i\beta)}\frac{J_0(\beta)\, J_1(\beta\xi)\,[1-(\beta\xi)^2] + J_1(\beta)\,\beta^2\xi J_1'(\beta\xi)}{2J_0^2(\beta)\,\beta\xi}\right. \tag{3.11}$$

$$\left. + 2i\sum_{n=1}^{\infty}\frac{J_1(\alpha_n\xi)\, J_1'(i\alpha_n\xi)\,\alpha_n^2}{J_1(\alpha_n)\, J_0(i\alpha_n)\,(\alpha_n^2-iR_m)^2} + iR_m\tau\left(\frac{J_1(i\beta\xi)}{\beta J_0(i\beta)}\right)^*\frac{J_1'(i\beta\xi)}{J_0(i\beta)}\right\}$$

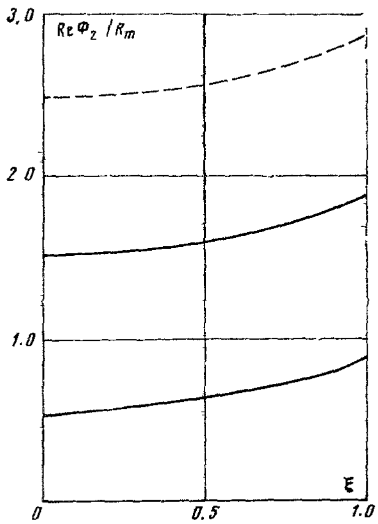

Fig. 2

As a consequence of replacing $\exp(-\alpha_n^2\tau)$ by zero the solution presented here is valid only for $\tau \gg 1$. Practically, it can be applied from the time $\tau = 1$ with only a small error.

For $R_m \ll 1$ the solution is simplified:

$$\operatorname{Re}\omega_1(r, \alpha, t > t_0) = \frac{H_0^2}{\pi\rho\nu_m}\frac{1}{\xi}\left\{\frac{R_m}{32}\xi^3\cos 2(\alpha - \Omega_0 t) + 2\sin\alpha\sum_{n=1}^{\infty}\frac{J_2(\alpha_n\xi)\, I_1(\alpha_n\xi)\,\alpha_n^2}{I_0(\alpha_n)\, J_1(\alpha_n)\,(\alpha_n^4 + R_m^2)}\right\}$$

$$\operatorname{Re}\omega_2(r, \alpha, t > t_0) = \frac{H_0^2}{4\pi\rho\nu_m}\frac{R_m}{\xi}\left\{\frac{\xi}{2}\left(\frac{1}{2} - \xi^2\right) + 16\sum_{n=1}^{\infty}\frac{J_1(\alpha_n\xi)\,[I_0(\alpha_n\xi) - (\alpha_n\xi)^{-1} I_1(\alpha_n\xi)]}{J_1(\alpha_n\xi)\, I_0(\alpha_n)\,\alpha_n^4} - \xi\tau\right\}$$

For a stationary cylinder ($\Omega_0 = 0$, $R_m = 0$) the vorticity ω_2 vanishes, and Re $\omega_1(r, \alpha, t > t_0)$ agrees with the result in [2]. The difference in constants arises from the use of different systems of units.

The dimensionless functions $\Phi_1(\xi)$ and $\Phi_2(\xi, \tau)$ were evaluated for three values of R_m (0.01, 0.1, 1.0); the results are presented graphically. Figure 1 shows the modulus (curve 1) and the argument (curve 2) of the function

$$\Phi_1(\xi) = |\Phi_1(\xi)|\, e^{-i[\pi/2 - \theta(\xi)]}$$

Since $|\Phi_1(\xi)|$ varies slowly with R_m, the three curves merge into one. The angle θ is nearly proportional to R_m, and therefore Fig. 1 shows $\theta(\xi)/R_m$, which is represented by a single curve for the three values of R_m.

Since the real value of $\omega_2(r, t)$ is completely determined by only the real part of $\Phi_2(\xi, \tau)$, Fig. 2 shows Re $\Phi_2(\xi, \tau)$ for two values of τ (1.0, 2.0) (solid curves). As a consequence of the linear dependence of $\Phi_2(\xi, \tau)$ on τ the function Re $\Phi_2(\xi, \tau)$ can be constructed from the data presented for any time τ. For clarity the curve for $\tau = 3$ is shown dashed in Fig. 2.

Equation (3.8) shows that the secondary flow under study is a superposition of two qualitatively different flows. The first of these, determined by the vorticity $\omega_1(r, \alpha, t)$ and containing the exponential factor exp $(2i\alpha)$, has a structure consisting of four separate cells. This flow is shown schematically in Fig. 3. The fluid circulates in opposite directions in adjacent cells; i.e., ω_1 changes sign in crossing the boundary between cells. Consequently, this boundary is determined by the condition Re $\omega_1(r, \alpha, t) = 0$. The equations $\alpha_0 = \alpha_0(\xi)$ of the radial parts of the boundaries of the cells, shown dashed in Fig. 3, take the form

$$\alpha_0 = -\frac{1}{2}\theta(\xi) + n\frac{\pi}{2} + \Omega_0 t$$

where $n = 0, 1, 2, 3$, respectively, for each of the four cells; the last term $\Omega_0 t$ ensures the rotation of the cells with constant angular velocity Ω_0.

In the general case $\Omega_0 \neq 0$ (i.e., $R_m \neq 0$) these boundaries are twisted. As can be seen from Fig. 1 the angle of torsion $\frac{1}{2}\theta(\xi)$ is directly proportional to R_m; for $R_m = 1$ this angle reaches 12°. It is clear from (3.10) that the "cellular" flow for $t > t_0$ ceases to depend on the time and is only shifted in space by the basic rotation with the angular velocity Ω_0.

The flow described by the vorticity $\omega_2(r, t)$ is completely axisymmetric. This axisymmetric flow continues to depend on time for $t > t_0$ and, consequently, the vorticity ω_2 and the flow velocity depend linearly on the time. This is a result of the fact that after the steady state is established the magnetic field continues to exert a strong effect ("resistance") on the rotating fluid and its effect is axisymmetric (3.7). As a result this flow becomes predominant for large t, and the secondary flow begins to approach axial symmetry. For the small values of R_m necessary for this the time turns out to be rather large, and the present treatment, which neglects viscous forces and the convective transport of eddies by secondary flow, becomes inapplicable earlier. If $R_m \gtrsim 1$, the flow corresponding to ω_2 rapidly becomes controlling. The velocity field will be determined later, and diagrams of the streamlines will be presented.

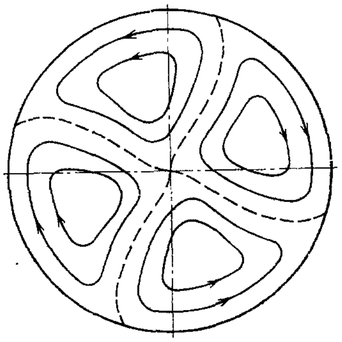

Fig. 3

4. The velocity field of the secondary flow is determined by the equations

$$\operatorname{div}\mathbf{v} = 0, \quad \operatorname{curl}\mathbf{v} = \boldsymbol{\omega} = \omega(r, \alpha, t)\,\mathbf{e}_z$$

from which it follows that

$$\mathbf{v} = \operatorname{curl}[\Psi(r, \alpha, t)\,\mathbf{e}_z] = \frac{1}{r}\frac{\partial\Psi}{\partial\alpha}\mathbf{e}_r - \frac{\partial\Psi}{\partial r}\mathbf{e}_\alpha \tag{4.1}$$

$$\Delta\Psi = -\omega = -\frac{H_0^2}{4\pi\rho\nu_m}\{\Phi_1(\xi)\, e^{2i(\alpha - R_m\tau)} + \Phi_2(\xi, \tau)\} \tag{4.2}$$

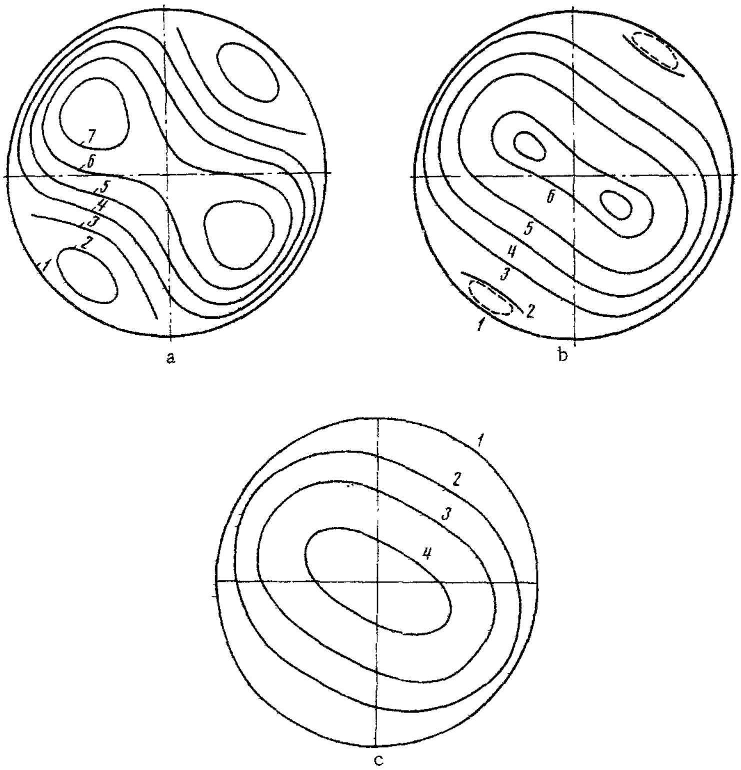

Fig. 4

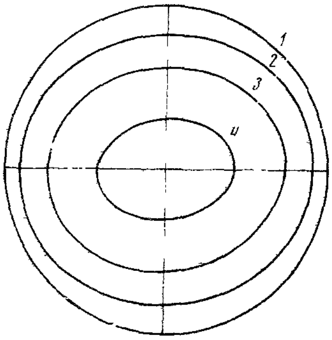

Fig. 5

On the right-hand side of the last equation the term $2\Omega_0$, which does not refer to secondary flow, has been omitted. The solution for the stream function is sought in the form

$$\Psi(r, \alpha, t) = \frac{H_0^2 a^2}{4\pi\rho\nu_m}\psi(\xi, \alpha, \tau), \quad \psi(\xi, \alpha, \tau) = \psi_1(\xi)\, e^{2i(\alpha - R_m\tau)} + \psi_2(\xi, \tau) \tag{4.3}$$

The functions $\psi_1(\xi)$ and $\psi_2(\xi, \tau)$ which are bounded at $\xi = 0$ are determined by the equations

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\psi_1}{d\xi}\right) - \frac{4}{\xi^2}\psi_1 = -\Phi_1(\xi) \tag{4.4}$$

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\psi_2}{d\xi}\right) = -\Phi_2(\xi, \tau) \tag{4.5}$$

and the supplementary boundary condition

$$\psi_1(1) = 0 \tag{4.6}$$

following from the condition $v_r|_{r=a} = 0$.

The function $\psi_2(\xi, \tau)$ can be determined to within a constant, since a constant does not affect the velocity v. For definiteness we use the condition $\psi_2(1, \tau) = 0$. The solution of the linear inhomogeneous equation (4.4) satisfying condition (4.6) is found by using the Green's function and has the form

$$\psi_1(\xi) = \frac{1}{4}\left\{\frac{1}{\xi^2}\int_0^{\xi} s^3\Phi_1(s)\,ds - \xi^2\int_0^1 s^3\Phi_1(s)\,ds + \xi^2\int_{\xi}^1 \frac{1}{s}\Phi_1(s)\,ds\right\}$$

Equation (4.5) can be integrated directly,

$$\psi_2(\xi, \tau) = \int_{\xi}^1 \frac{1}{p}\int_0^p s\Phi_2(s, \tau)\,ds\,dp = -\ln\xi\int_0^{\xi} s\Phi_2(s, \tau)\,ds - \int_{\xi}^1 (\ln s)\, s\Phi_2(s, \tau)\,ds$$

The dimensionless stream function $\psi(\xi, \alpha, \tau)$ determines the velocity field through (4.1) and (4.3),

$$\mathbf{v}(r, \alpha, t) = \frac{H_0^2 a}{4\pi\rho\nu_m}\left[\frac{1}{\xi}\operatorname{Re}\frac{\partial\psi}{\partial\alpha}\mathbf{e}_r - \operatorname{Re}\frac{\partial\psi}{\partial\xi}\mathbf{e}_\alpha\right] \tag{4.7}$$

The streamlines are represented by the lines for Re $\psi(\xi, \alpha, \tau) = \text{const}$.

Figures 4 and 5 show the streamlines with the corresponding values of the dimensionless stream function ψ for $R_m = 0.1$ and $R_m = 1.0$. The change in the character of the secondary flow with time is traced for $R_m = 0.1$. For $\tau = 1$ (Fig. 4a, curves 1-7 correspond to the values $\psi = 0$, 0.005, 0, −0.005, −0.01, −0.015, and −0.02, respectively) the flow is similar to that found in [2] when the field penetrates a stationary cylinder. There are cells with opposite directions of rotation although they do not reach the center of the circle, since close to the center ω_2 predominates over ω_1 even for $\tau = 1$, as can be seen from Figs. 1 and 2.

For $\tau = 2$ (Fig. 4b, curves 1-6 correspond to $\psi = 0$, 0, −0.01, −0.02, −0.03, and −0.04, respectively) axisymmetric flow corresponding to the vorticity $\omega_2(r, t)$ is not so intense as to completely smear out the cellular structure of the flow corresponding to the vorticity $\omega_1(r, \alpha, t)$. Only for $\tau = 3$ (Fig. 4c, curves 1-4 correspond to $\psi = 0$, −0.02, − 0.04, and −0.06, respectively) does the flow take on the second characteristic form with the streamlines encircling the axis of the cylinder.

For $R_m = 1$ the flow $\omega_2(r, t)$ predominates from the time $\tau = 1$, and therefore on Fig. 5 the streamlines are shown only for this instant. Here 1) $\psi = 0$; 2) $\psi = -0.04$; 3) $\psi = -0.09$; 4) $\psi = -0.14$.

On the basis of (4.7) assumption (1.1) is equivalent to the condition

$$(H_0^2/4\pi\rho)\,(4\pi\sigma\; a/c^2)^2 < 1$$

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